

Analysis of Quantum Network Coding for Realistic Repeater Networks

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Quantum repeater networks have attracted attention for the implementation of long-distance and large-scale sharing of quantum states. Recently, researchers extended classical network coding, which is a technique for throughput enhancement, into quantum information. The utility of quantum network coding (QNC) has been shown under ideal conditions, but it has not been studied previously under conditions of noise and shortage of quantum resources. We analyzed QNC on a butterfly network, which can create end-to-end Bell pairs at twice the rate of the standard quantum network repeater approach. The joint fidelity of creating two Bell pairs has a small penalty for QNC relative to entanglement swapping. It will thus be useful when we care more about throughput than fidelity. We found that the output fidelity drops below 0.5 when the initial Bell pairs have fidelity $F < 0.90$, even with perfect local gates. Local gate errors have a larger impact on quantum network coding than on entanglement swapping.

I. INTRODUCTION

Researchers are striving to produce quantum communication technology for long-range transmission of quantum information and sharing of distributed quantum states [1–3]. Quantum information requires a network specialized for quantum communication. Quantum information may enable new functions not achievable using classical information. For example, quantum key distribution creates a shared random sequence of bits between two parties [4, 5]. Because quantum information cannot in general be measured without disturbing the state and cannot be cloned [6], statistical tests can prove the absence of an eavesdropper, guaranteeing the secrecy of the bit values. QKD technology is already realized at a commercial level for urban scale, complex topology networks [7, 8].

Besides QKD, other distributed security functions [9], general purpose distributed quantum computing and blind quantum computing [10] have been proposed as uses of long distance quantum communication. In addition, the realization of inter-continental and inter-major city QKD is also desired.

Thus, there is a growing need for large-scale quantum networks, but the current quantum network protocol suffers from a distance limit set by the probability of correctly receiving a photon through an exponentially lossy channel and other factors. In order to solve this problem, quantum repeaters have been proposed [11] and many of the components have been experimentally demonstrated [12, 13]. A quantum repeater has multiple important roles: to create and share physical en-

tanglement pairs (Bell pairs) between nearest neighbors over short distances, to perform purification of Bell pairs, and to create one long Bell pair by connecting two entangled pairs using entanglement swapping [11, 14–19]. Long range, complex quantum networks can be realized by arranging a number of quantum repeaters and links. However, the cost of quantum communication per unit of quantum information (e.g. qubit) is very high compared with classical communication.

Quantum network coding (QNC) may contribute to solving this problem. Network coding [20] is known as a bottleneck elimination method in classical networks. For example, Fig. 1 shows simultaneous transmission over the directed classical butterfly network using network coding. Two bits can be sent in one use of each link even though each individual transmission would result in conflicts for access to individual links. The butterfly network is the simplest case showing a throughput bottleneck which can be alleviated using quantum network coding. Verifying the behavior on this graph can show that quantum network coding can give an advantage over simple routing schemes in some circumstances. It is expected that network coding also allows the same resolution in a quantum network. In recent years, a number of researchers have studied quantum network coding [21–27]. However, all of these studies presuppose the use of pure states and perfect local gates. The effects of errors and resource shortages are unknown. In this paper, we aim to determine the usefulness of quantum network coding using mixed states.

First, we assume Pauli errors on the Bell pairs that are our initial resources. We investigate the error propagation in the QNC procedure and calculate the change of fidelities step by step in our coding scheme. These calculations enable us to compare the communication efficiency between QNC and entanglement swapping as used in many quantum repeater designs. Furthermore, we calculate error thresholds for practical QNC on the butterfly

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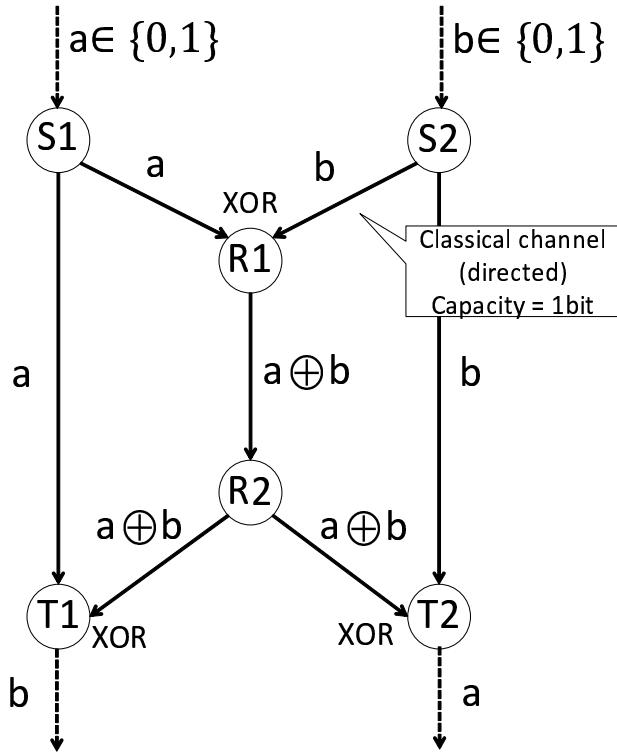


FIG. 1. The classical network coding scheme on the butterfly network. The problem is to send a bit of information a from sender node $S1$ to target node $T2$ and b from $S2$ to $T1$ simultaneously. It is clearly impossible to solve this problem using simply routing. XOR operations on relay node $R1$ and target nodes $T1, T2$ solve this problem.

graph and find that initial resource fidelities are required to be $F \geq 0.9$ to achieve the final fidelity over 0.5.

Next, we assume Pauli errors on every CNOT gate, single qubit rotation, measurement, and quantum memory storage time step and calculate the final fidelities using Monte Carlo simulation to assess the complete protocol.

The rest of this paper is organized as follows. In Section II, we show the protocol and related matters of quantum network coding for quantum repeaters. In Section III, we present the analysis of quantum network coding and entanglement swapping scheme in the presence of X and Z errors. In Section V, we conclude the discussion of this paper.

II. QUANTUM NETWORK CODING

Let us review the concept of quantum network coding for quantum repeaters by examining the butterfly graph in Fig. 2 [28]. Quantum network coding, like classical network coding, shifts the location of required communication away from the single bottleneck link, to other links in the network, reducing demand on the bottleneck link. We assume that the performance of all links is the

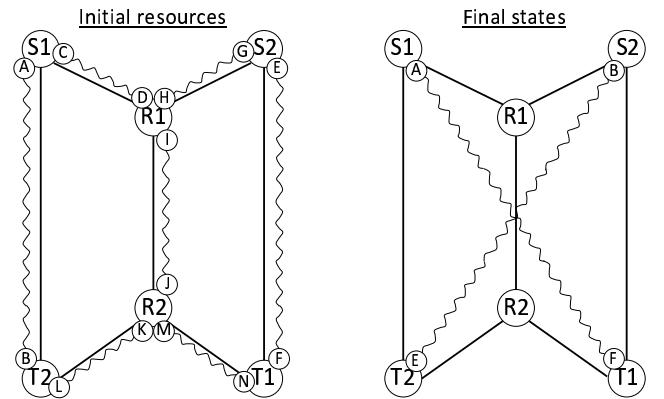


FIG. 2. Initial resources and final states for QNC. Each repeater node contains two or three qubits entangled with neighbors into Bell pairs as shown. Our goal is to establish Bell pairs between nodes $S1$ & $T1$, and $S2$ & $T2$.

same, and that the number of times that the most-used link must be used to complete our operation determines our ultimate performance. We begin with $|\Psi^+\rangle$ Bell pairs across the seven links as shown. In this section, we use the ket vector notation to describe the pure state with fidelity $F = 1$. In following sections, we will use the ket vector to describe mixed states, as discussed in the beginning of section III.

A. Encoding operations

To describe the QNC protocol, we first introduce the following three encoding operators. They consist of CNOT gate operations, \hat{Z} basis measurement operators, and one qubit rotation based on measurement results. The CNOTs are executed between a Control bit and a Bell pair, where we designate one member of the Bell pair the Resource qubit, and the other the Target qubit. The Control qubit $C(C_1, C_2)$ and the Resource qubit $R(R_1, R_2)$ exist on the same repeater. An \hat{X} or \hat{Z} rotation is performed on the Target qubit $T(T_1, T_2)$ if and only if the measurement result is positive. Our operations are

$$\mathbf{Con}_{R \rightarrow T}^C = \hat{X}_T^{S_1} \hat{P}_{\hat{Z}, R}^{\pm, S_1} \mathbf{CNOT}^{(C, R)} \quad (1)$$

$$\mathbf{Add}_{R \rightarrow T}^{C_1, C_2} = \hat{X}_T^{S_1} \hat{P}_{\hat{Z}, R}^{\pm, S_1} \mathbf{CNOT}^{(C_2, R)} \mathbf{CNOT}^{(C_1, R)} \quad (2)$$

$$\begin{aligned} \mathbf{Fanout}_{R_1 \rightarrow T_1, R_2 \rightarrow T_2}^C &= \hat{X}'_{T_2} \hat{X}_{T_1}^{S_2} \hat{P}_{\hat{Z}, R_1}^{\pm, S_2} \hat{P}_{\hat{Z}, R_2}^{\pm, S_1} \\ &\quad \mathbf{CNOT}^{(C, R_2)} \mathbf{CNOT}^{(C, R_1)} \end{aligned} \quad (3)$$

where \hat{P}^{\pm} is the projective measurement operator

$$\hat{P}_{\hat{X}}^{\pm} = \frac{1}{2} (\mathbf{1} \pm \hat{X}), \quad \hat{P}_{\hat{Z}}^{\pm} = \frac{1}{2} (\mathbf{1} \pm \hat{Z}), \quad (4)$$

\hat{X} and \hat{Z} are the normal Pauli operators, and S_1 and S_2 are measurement outcomes of the operator $\hat{P}_{\hat{X}}^{\pm}$ and $\hat{P}_{\hat{Z}}^{\pm}$.

These operations correspond to the bit transfer, add and fanout operations in a classical network coding protocol [20]. Fig. 3 shows quantum circuits for $\mathbf{Con}_{B \rightarrow C}^A$, $\mathbf{Add}_{F \rightarrow G}^{D,E}$, and $\mathbf{Fanout}_{I \rightarrow J, K \rightarrow L}^H$.

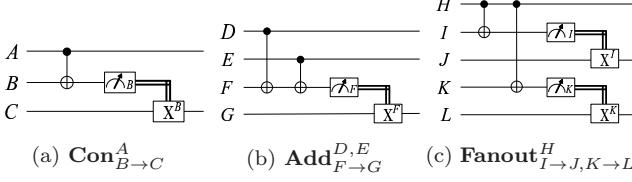


FIG. 3. (a) Connection operation between the qubit A and the Bell pair BC . (b) Add operation between qubit D, E , and the Bell pair FG . (c) Fanout operation between the qubit H , and the Bell pair IJ and KL .

B. Removal operations

We also introduce the following two removal operators. These operators are unique to quantum network coding protocols, because we have to remove unnecessary entangled qubits before the end of the procedure. To remove these qubits without causing changes on the remaining system, we use \hat{X} basis measurements and feedforward operations based on the measurement results. Our operations are

$$\mathbf{Rem}_{R \rightarrow T} = \hat{Z}_T^{S_1} \hat{P}_{\hat{X}, R}^{\pm, S_1} \quad (5)$$

$$\mathbf{RemAdd}_{R \rightarrow T_1, T_2} = \hat{Z}_{T_2}^{S_2} \hat{Z}_{T_1}^{S_2} \hat{P}_{\hat{X}, R}^{\pm, S_2}. \quad (6)$$

Rem removes the qubits used as target qubits in the connection and fanout operations, and **RemAdd** removes the qubits used as target qubits in the add operations in QNC protocol.

C. QNC

Here, we introduce the protocol operator **QNC** to describe the complete procedure for QNC. All operations in this procedure are LOCC as shown above.

$$\mathbf{QNC}|\psi_{init}^{\mathbf{QNC}}\rangle = \mathbf{Rem}_{H \rightarrow E} \mathbf{Rem}_{D \rightarrow A} \mathbf{RemAdd}_{J \rightarrow D, H} \mathbf{Rem}_{N \rightarrow J} \mathbf{Rem}_{L \rightarrow J} \mathbf{CNOT}^{(L, B)} \quad (7)$$

$$\mathbf{CNOT}^{(N, F)} \mathbf{Fanout}_{K \rightarrow L, M \rightarrow N}^J$$

$$\mathbf{Add}_{I \rightarrow J}^{D, H} \mathbf{Con}_{G \rightarrow H}^E \mathbf{Con}_{C \rightarrow D}^A |\psi_{init}\rangle \quad (7)$$

$$= |\psi_{final}^{\mathbf{QNC}}\rangle = |\Psi^+\rangle_{AF} \otimes |\Psi^+\rangle_{BE}. \quad (8)$$

Here,

$$|\psi_{init}^{\mathbf{QNC}}\rangle = |\Psi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD} \otimes |\Psi^+\rangle_{EF} \otimes |\Psi^+\rangle_{GH} \otimes |\Psi^+\rangle_{IJ} \otimes |\Psi^+\rangle_{KL} \otimes |\Psi^+\rangle_{MN}. \quad (9)$$

When we perform **QNC** on the seven Bell pairs, we can create two crossed Bell pairs as a result. In this state, we can perform quantum teleportation between repeaters in opposite corners simultaneously, as shown in Fig. 2. The total circuit of **QNC** is shown in Fig. 13.

D. QNC versus entanglement swapping

To compare this QNC protocol with the existing repeater protocols, we also introduce the protocol operator **2ES**. In this procedure, we perform two entanglement swapping operations using three Bell pairs.

$$\mathbf{2ES}|\psi_{init}^{\mathbf{2ES}}\rangle = \mathbf{ES}_{(M, N)}^{(C, J)} \mathbf{ES}_{(I, J)}^{(C, D)} |\psi_{init}^{\mathbf{2ES}}\rangle \quad (10)$$

$$= |\psi_{final}^{\mathbf{2ES}}\rangle = |\Psi^+\rangle_{CN}. \quad (11)$$

Here,

$$\mathbf{ES}_{(I, J)}^{(C, D)} = \mathbf{Rem}_{D \rightarrow C} \mathbf{Con}_{I \rightarrow J}^D \quad (12)$$

$$|\psi_{init}^{\mathbf{2ES}}\rangle = |\Psi^+\rangle_{CD} \otimes |\Psi^+\rangle_{IJ} \otimes |\Psi^+\rangle_{MN}. \quad (13)$$

Entanglement swapping between two Bell pairs can generate one long Bell pair [19]. **Rem** removes the leftover qubit for this operation.

Next, we discuss the bottleneck problem on the butterfly network. In this case, we cannot perform **2ES** two times and share two target Bell pairs between AF and BE without remaking Bell pairs as shown in Fig. 4. Bell pair IJ is the bottleneck limiting the performance. One

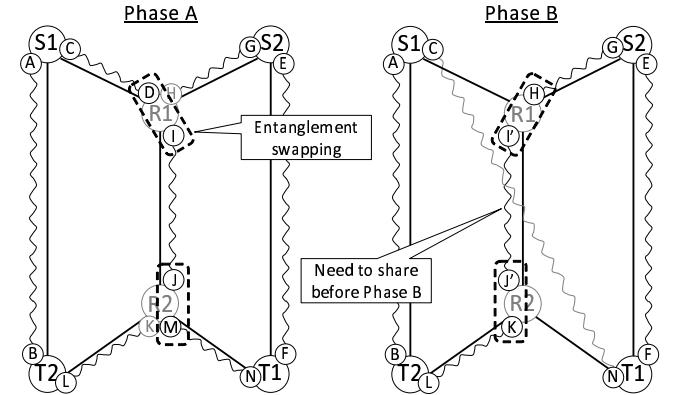


FIG. 4. Conceptual diagram of communication using entanglement swapping. Simultaneous execution of Phase A and Phase B is not possible. Re-sharing of a Bell-pair is needed between $R1$ and $R2$. The AB and EF Bell pairs are unused in this protocol.

approach to solving this bottleneck problem is link multiplexing [29]. In this scheme, an approach such as time division multiplexing is proposed to solve the bottleneck problem on a dumbbell network with few shared Bell pairs. To compare **2ES** and network coding, we adopt this scheme. Note that network coding generates the two goal Bell pairs while consuming seven Bell pairs in one

cycle, whereas entanglement swapping consumes only six Bell pairs but requires two cycles because of the resource conflict. When we assume the time necessary to share Bell pairs between nearest neighbor repeaters and the memory lifetime of Bell pairs are similar, it is hard to share extra Bell pairs between bottleneck repeaters.

III. ERRORS ON THE INITIAL BELL PAIRS

To elucidate the advantage of QNC, if any, we compare the communication fidelity of QNC and 2ES. Before tackling the more general problem including gate errors, we investigate the propagation of X and Z errors present in the initial seven Bell pairs in Fig. 2. We define $\epsilon_{qubit, \hat{X}(\hat{Z})}$ as $\hat{X}(\hat{Z})$ rotation error with probability p . Due to the symmetry of Bell pairs, we do not need to distinguish between an error on qubit A and one on qubit B . For example, we describe those two types of errors on Bell pair $|\Psi^+\rangle_{AB}$ as follows:

$$\epsilon_{A, \hat{X}} |\Psi^+\rangle_{AB} = F |\Psi^+\rangle_{AB} + (1 - F) |\Phi^+\rangle_{AB} \quad (14)$$

$$\epsilon_{A, \hat{Z}} |\Psi^+\rangle_{AB} = F |\Psi^+\rangle_{AB} + (1 - F) |\Psi^-\rangle_{AB}. \quad (15)$$

Here, fidelity $F = 1 - p = \langle \psi | \rho | \psi \rangle$ where $|\psi\rangle$ is the desired pure state. In this paper, for simplicity of representation, we retain the ket notation even for mixed states. The above should be understood to represent

$$\rho = \sqrt{\epsilon_{A, \hat{X}}} |\Psi^+\rangle \langle \Psi^+|_{AB} \sqrt{\epsilon_{A, \hat{X}}} \quad (16)$$

$$= F |\Psi^+\rangle \langle \Psi^+|_{AB} + (1 - F) |\Phi^+\rangle \langle \Phi^+|_{AB}. \quad (17)$$

In this section, we assume that we can perform single qubit rotation, CNOT gate, and projective measurement perfectly with success probability 1. Gate errors will be incorporated in Sec. IV.

A. Z errors

Here, we discuss Z errors on our initially shared Bell pairs. Z errors propagate via a CNOT gate from target qubit to control qubit.

1. Connection

First, we investigate the Z error propagation in the Connection operation. When we perform Connection $\mathbf{Con}_{C \rightarrow D}^B$ between Bell pairs AB and CD with probabilistic Z errors on qubits A and C, the Z error on measured qubit C causes a similar error on qubit B. Then,

the initial state $|\psi'^{\mathbf{Con}}_{init}\rangle$ can be described as follows:

$$|\psi'^{\mathbf{Con}}_{init}\rangle = \epsilon_{A, \hat{Z}}^{(A)} |\Psi^+\rangle_{AB} \otimes \epsilon_{C, \hat{Z}}^{(C)} |\Psi^+\rangle_{CD}. \quad (18)$$

After the Connection operation, the final state $\epsilon_{B, \hat{Z}}^{(C)} \epsilon_{A, \hat{Z}}^{(A)} |\psi'^{\mathbf{Con}}_{final}\rangle$ becomes

$$|000\rangle_{ABD} + \sum_{S_{AB}}^{0,1} \sum_{S_{CD}}^{0,1} p_{S_{AB}} p_{S_{CD}} (-1)^S |111\rangle_{ABD}. \quad (19)$$

Here, $\epsilon_{Q, \hat{Z}}^{(P)}$ denotes a Z error on qubit Q resulting from the original Z error on qubit P . S is calculated as follows:

$$S = S_{AB} + S_{CD}. \quad (20)$$

S_{AB} and S_{CD} are 1 if the corresponding Bell pair includes a Z error, otherwise they are 0. When we assume the initial fidelity of each Bell pair $F_{AB} = F_{CD} = F$, the result is a phase flip error ($S = 1$) with probability $2F(1 - F)$, otherwise $S = 0$. We show pre-operation and post-operation fidelities in Fig. 5.

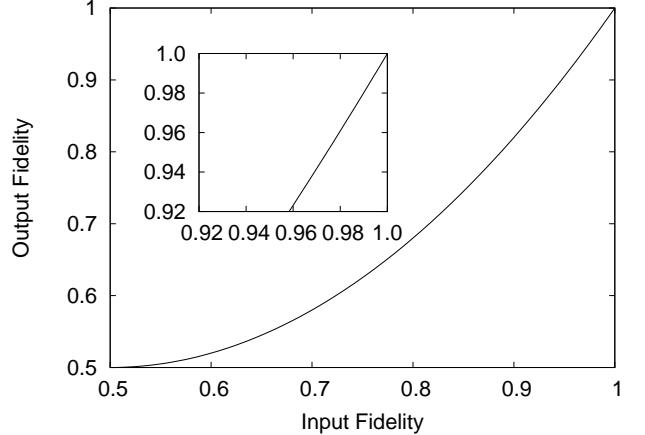


FIG. 5. Fidelity against Bell pair Z errors only during the Connection operation. The horizontal axis corresponds to the initial fidelity of each Bell pair. The vertical axis corresponds to the final fidelity of the system. Local gates are assumed to be perfect.

2. Add

Second, we investigate the error propagation in the Add operation. For example, we perform $\mathbf{Add}_{I \rightarrow J}^{F, H}$ with three Bell pairs EF , GH , and IJ . The initial state $|\psi'^{\mathbf{Add}}_{init}\rangle$ can be described as follows:

$$|\psi'^{\mathbf{Add}}_{init}\rangle = \epsilon_{I, \hat{Z}}^{(I)} \epsilon_{G, \hat{Z}}^{(G)} \epsilon_{E, \hat{Z}}^{(E)} |\Psi^+\rangle_{EF} \otimes |\Psi^+\rangle_{GH} \otimes |\Psi^+\rangle_{IJ}. \quad (21)$$

After the Add operation, the final state $\epsilon_{H, \hat{Z}}^{(I)} \epsilon_{F, \hat{Z}}^{(I)} \epsilon_{G, \hat{Z}}^{(G)} \epsilon_{E, \hat{Z}}^{(E)} |\psi'^{\mathbf{Add}}_{final}\rangle$ becomes

$$\sum_{S_{AB}}^{\{0,1\}} \sum_{S_{EF}}^{\{0,1\}} \sum_{S_{GH}}^{\{0,1\}} p_{S_{AB}} p_{S_{EF}} p_{S_{GH}} (|0000\rangle + (-1)^{S_0} |1111\rangle)_{EFGH} |0\rangle_J + (-1)^{S_1} (|0011\rangle + (-1)^{S_0} |1100\rangle)_{EFGH} |1\rangle_J. \quad (22)$$

Here, $|\psi_{final}^{\text{Add}}\rangle$ corresponds to the state in Eq. 2. Each S_i is calculated as follows:

$$S_0 = S_{AB} + S_{EF}, \quad (23)$$

$$S_1 = S_{EF} + S_{IJ}. \quad (24)$$

When all $S_i \neq 1$ where $i \in \{0, 1\}$, which occurs with probability $F^3 + (1 - F)^3$, then the final state is error free.

3. Fanout

Third, we investigate the error propagation in the Fanout operation. When we perform $\text{Fanout}_{K \rightarrow L, M \rightarrow N}^J$

with three Bell pairs IJ , KL and MN , the initial state $|\psi_{init}^{\text{Fanout}}\rangle$ can be described as follows:

$$|\psi_{init}^{\text{Fanout}}\rangle = \epsilon_{I, \hat{Z}}^{(I)} \epsilon_{K, \hat{Z}}^{(K)} \epsilon_{M, \hat{Z}}^{(M)} |\Psi^+\rangle_{IJ} \otimes |\Psi^+\rangle_{KL} \otimes |\Psi^+\rangle_{MN}. \quad (25)$$

After the Fanout operation, the final state $\epsilon_{I, \hat{Z}}^{(I)} \epsilon_{J, \hat{Z}}^{(K)} \epsilon_{J, \hat{Z}}^{(M)} |\psi_{final}^{\text{Fanout}}\rangle$ becomes

$$\sum_{S_{IJ}}^{\{0,1\}} \sum_{S_{KL}}^{\{0,1\}} \sum_{S_{MN}}^{\{0,1\}} p_{S_{IJ}} p_{S_{KL}} p_{S_{MN}} (|00\rangle_{LN} + (-1)^{S_0} |11\rangle_{LN}) |0\rangle_J + (-1)^{S_1} (|01\rangle_{LN} + (-1)^{S_0} |10\rangle_{LN}) |1\rangle_J. \quad (26)$$

Here, $|\psi_{final}^{\text{Fanout}}\rangle$ corresponds to the state in Eq. 3. Each S_i is calculated as follows

$$S_0 = S_{EF} + S_{GH}, \quad (27)$$

$$S_1 = S_{GH} + S_{IJ}. \quad (28)$$

When all $S_i = 0$ where $i \in \{0, 1\}$, which occurs with probability F^3 , then the final state is error free.

4. Removal and Removal-Add

In Removal and Removal-Add operation, we perform X basis measurement on the target qubit. When a Z error exists on the target qubit, the measurement result flips. Removal and Removal-Add move a Z error from the measured qubit to the feedforward qubit(s). We show this error propagation below:

$$\text{Rem}_{R \rightarrow T} \epsilon_{R, \hat{Z}}^{(R)} |\psi_{init}^{\text{Rem}}\rangle = \epsilon_{T, \hat{Z}}^{(R)} |\psi_{final}^{\text{Rem}}\rangle, \quad (29)$$

$$\text{RemAdd}_{R \rightarrow T_1, T_2} \epsilon_{R, \hat{Z}}^{(R)} |\psi_{init}^{\text{RemAdd}}\rangle = \epsilon_{T_1, \hat{Z}}^{(R)} \epsilon_{T_2, \hat{Z}}^{(R)} |\psi_{final}^{\text{RemAdd}}\rangle. \quad (30)$$

To conclude the above discussion, we show the location of errors which cause Z errors on final Bell pairs in Fig. 6.

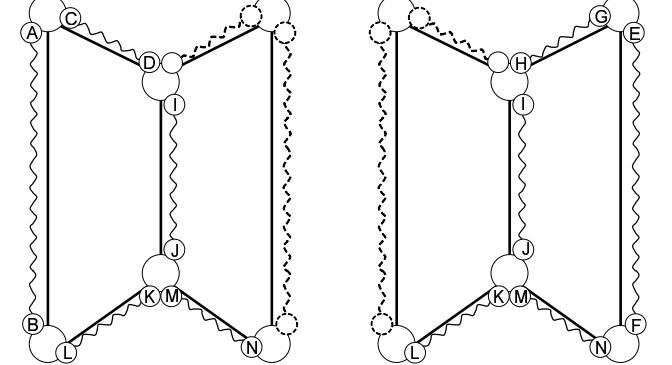


FIG. 6. Z errors propagation. The left figure shows the five Bell pairs that affect the final Bell pair AF. The right figure shows the five Bell pairs that affect the final Bell pair BE.

5. Comparison

To compare QNC and 2ES, we first calculate the final fidelity after the complete QNC sequence. When we assume each initial Bell pair has a Z error on one qubit with probability $1 - F$, the initial state $|\psi_{init}^{\text{QNC}}\rangle$ and final

state $|\psi'_{final}^{\text{QNC}}\rangle$ become

$$|\psi'_{init}^{\text{QNC}}\rangle = \epsilon_{M,\hat{Z}}^{(M)} \epsilon_{K,\hat{Z}}^{(K)} \epsilon_{I,\hat{Z}}^{(I)} \epsilon_{G,\hat{Z}}^{(G)} \epsilon_{E,\hat{Z}}^{(E)} \epsilon_{C,\hat{Z}}^{(C)} \epsilon_{A,\hat{Z}}^{(A)} |\psi_{init}^{\text{QNC}}\rangle, \quad (31)$$

$$|\psi'_{final}^{\text{QNC}}\rangle = \sum_m^0 \sum_n^0 P_{m,n} \hat{Z}_A^m \hat{Z}_B^n |\psi_{final}^{\text{QNC}}\rangle. \quad (32)$$

where m and n are the absence (0) or presence (1) of Z errors on the final *AF* and *BE* Bell pairs, respectively (or equivalently on the *A* and *B* qubits after use of the Bell pairs for e.g. teleportation). The probability of each case $P_{m,n}$ is

$$P_{0,0} = F^7 + 5F^5(1-F)^2 + 12F^4(1-F)^3 + 7F^3(1-F)^4 + 4F^2(1-F)^5 + 3F(1-F)^6, \quad (33)$$

$$P_{0,1} = P_{1,0} = 2F^6(1-F) + 6F^5(1-F)^2 + 8F^4(1-F)^3 + 8F^3(1-F)^4 + 6F^2(1-F)^5 + 2F(1-F)^6, \quad (34)$$

$$P_{1,1} = 3F^6(1-F) + 4F^5(1-F)^2 + 7F^4(1-F)^3 + 12F^3(1-F)^4 + 5F^2(1-F)^5 + (1-F)^7. \quad (35)$$

Each of the 128 combinations in Fig. 7 occurs with probability $F^{(7-w)}(1-F)^w$ where w is the Hamming weight of the bitstring.

0000000	0010000	0100000	0110000	1000000	1010000	1100000	1110000
0000001	0010001	0100001	0110001	1000001	1010001	1100001	1110001
0000010	0010010	0100010	0110010	1000010	1010010	1100010	1110010
0000011	0010011	0100011	0110011	1000011	1010011	1100011	1110011
0000100	0010100	0100100	0110100	1001000	1010100	1100100	1110100
0000101	0010101	0100101	0110101	1001001	1010101	1100101	1110101
0000110	0010110	0100110	0110110	1000110	1010110	1100110	1110110
0000111	0010111	0100111	0110111	1000111	1010111	1100111	1110111
0001000	0011000	0101000	0111000	1001000	1011000	1101000	1111000
0001001	0011001	0101001	0111001	1001001	1011001	1101001	1111001
0001010	0011010	0101010	0111010	1001010	1011010	1101010	1111010
0001011	0011011	0101011	0111011	1001011	1011011	1101011	1111011
0001100	0011100	0101100	0111100	1001100	1011100	1101100	1111100
0001101	0011101	0101101	0111101	1001101	1011101	1101101	1111101
0001110	0011110	0101110	0111110	1001110	1011110	1101110	1111110
0001111	0011111	0101111	0111111	1001111	1011111	1101111	1111111

FIG. 7. Chart of Z errors. Each seven-bit string indicates the presence (1) or absence (0) of a Z error on the Bell pairs *AB*, *CD*, .., and *LM*, respectively. The style of the string corresponds to the error existence on final state. Black and roman means no error, gray(italic) means Z error on *AF*(*BE*), and gray and italic means errors on both Bell pairs.

Next, we calculate the final fidelity in the 2ES scheme. When we assume each Bell pair for initial resource has a Z error on one qubit with probability $1-F$, the initial state $|\psi'_{init}^{\text{2ES}}\rangle$ and final state $|\psi'_{final}^{\text{2ES}}\rangle$ become as follows:

$$|\psi'_{init}^{\text{2ES}}\rangle = \epsilon_{M,\hat{Z}}^{(M)} \epsilon_{I,\hat{Z}}^{(I)} \epsilon_{C,\hat{Z}}^{(C)} |\psi_{init}^{\text{2ES}}\rangle, \quad (36)$$

$$|\psi'_{final}^{\text{2ES}}\rangle = \sum_m^0 P_m \hat{Z}_A^m |\Psi^+\rangle_{CN}. \quad (37)$$

Here, we show the probability of each case P_m below:

$$P_0 = 1F^3 + 3F(1-F)^2, \quad (38)$$

$$P_1 = 3F^2(1-F) + (1-F)^3. \quad (39)$$

We show the relationship between the input fidelity and the output fidelity of our network coding protocol and 2-entanglement swapping in Fig. 8. Here, the final

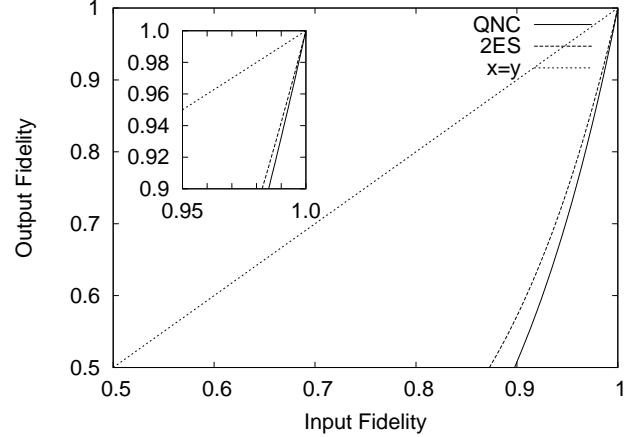


FIG. 8. Comparison of Swapping and QNC with Z errors only. Both show a substantial penalty compared to the fidelity of a single Bell pair (the $x = y$ line).

state with $F_{output} < 0.5$ has no practical use for quantum communication. When $F_{input} \leq 0.87$, the 2ES protocol falls below $F_{out} = 0.5$. When $F_{input} \leq 0.9$, the QNC protocol also falls below $F_{out} = 0.5$.

B. Classical correlation

Next, we discuss the classical correlation between two final Bell states. When we assume the input fidelity $F = 0.90$, the probability of the possible resulting states of both the *AF* and *BE* Bell pairs is shown in Table I by the formula (35). The correlation coefficient is

	$ \Psi_{BE}^+\rangle$	$ \Psi_{BE}^-\rangle$	
$ \Psi_{AF}^+\rangle$	a 0.516	b 0.148	e 0.664
$ \Psi_{AF}^-\rangle$	c 0.148	d 0.189	f 0.336
	g 0.664	h 0.336	

TABLE I. The correlation between $|\Psi_{AF}\rangle$ and $|\Psi_{BE}\rangle$ for input fidelity $F = 0.9$, Z errors only, and perfect local gates.

$$\phi = \frac{ad - bc}{\sqrt{efgh}} = 0.339. \quad (40)$$

The two output Bell pairs are unentangled using this error model but their error probabilities are classically correlated. This correlation is weak, despite the overlap of three Bell pairs in the left and right halves of Fig. 6.

C. X errors

Next, we discuss X errors on the initially shared Bell pairs. X errors propagate via CNOT gate from control qubit to target qubit.

1. Connection

First, we investigate the error propagation in Connection, when we perform Connection $\mathbf{Con}_{C \rightarrow D}^B$ between Bell pairs AB and CD with probabilistic X errors on qubits B and D. The initial state $|\psi''_{init}^{\mathbf{Con}}\rangle$ can be described as follows:

$$|\psi''_{init}^{\mathbf{Con}}\rangle = \epsilon_{D, \hat{X}}^{(D)} \epsilon_{B, \hat{X}}^{(B)} |\Psi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD}. \quad (41)$$

After the Connection operation, the final state $\epsilon_{D, \hat{X}}^{(D)} \epsilon_{B, \hat{X}}^{(B)} |\psi_{final}^{\mathbf{Con}}\rangle$ becomes

$$\sum_{S_{AB} S_{CD}}^{0,1} \sum_{S_{AB} S_{CD}}^{0,1} p_{S_{AB}} p_{S_{CD}} \hat{X}_A^{S_{AB}} \hat{X}_D^{S_{CD}} (|000\rangle + |111\rangle)_{ABD}. \quad (42)$$

Here, $\epsilon_{Q, \hat{X}}^{(P)}$ denotes an X error on qubit Q from the original X error on qubit P .

When we assume the initial Fidelity of each Bell pair $F_{AB} = F_{CD} = F$, each $S_i = 1$ with probability $2F(1 - F)$, otherwise it is 0. The fidelities of the input and output states in the Connection operation are plotted in Fig. 9.

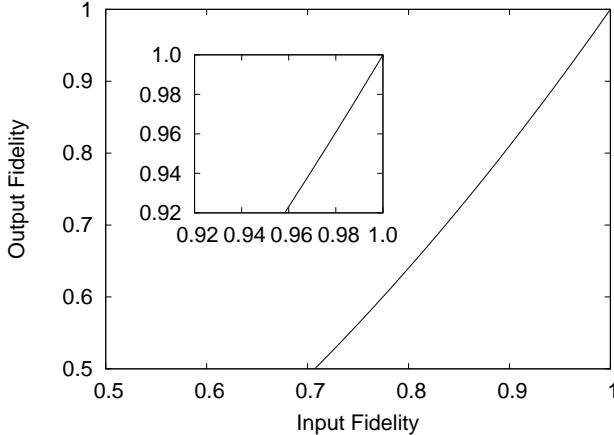


FIG. 9. Fidelity against X errors during Connection.

2. Add

Second, we investigate the X error propagation in the Add operation. When we perform $\mathbf{Add}_{I \rightarrow J}^{F, H}$ to three X error included Bell pairs $|\Psi^+\rangle_{EF}$, $|\Psi^+\rangle_{GH}$, and $|\Psi^+\rangle_{IJ}$, the initial state $|\psi''_{init}^{\mathbf{Add}}\rangle$ and the final state $|\psi''_{final}^{\mathbf{Add}}\rangle$ can be described as follows:

$$|\psi''_{init}^{\mathbf{Add}}\rangle = \epsilon_{I, \hat{X}}^{(I)} \epsilon_{G, \hat{X}}^{(G)} \epsilon_{E, \hat{X}}^{(E)} |\Psi^+\rangle_{EF} \otimes |\Psi^+\rangle_{GH} \otimes |\Psi^+\rangle_{IJ}. \quad (43)$$

After the Add operation, the final system $\epsilon_{J, \hat{X}}^{(I)} \epsilon_{G, \hat{X}}^{(G)} \epsilon_{E, \hat{X}}^{(E)} |\psi_{final}^{\mathbf{Add}}\rangle$ becomes

$$\sum_{S_{EF}}^{0,1} \sum_{S_{GH}}^{0,1} \sum_{S_{IJ}}^{0,1} p_{S_{EF}} p_{S_{GH}} p_{S_{IJ}} \hat{X}_J^{S_{IJ}} \hat{X}_G^{S_{GH}} \hat{X}_E^{S_{EF}} ((|0000\rangle + |1111\rangle)_{EFGH}|0\rangle_J + (|0011\rangle + |1100\rangle)_{EFGH}|1\rangle_J). \quad (44)$$

When all Bell pairs' fidelities are equal, the final state's fidelity becomes F^3 .

3. Fanout

Third, we investigate the X error propagation in Fanout operation. When we perform $\mathbf{Fanout}_{M \rightarrow N, O \rightarrow P}^L$

with three Bell pairs KL , MN and OP . Initial state $|\psi''_{init}^{\text{Fanout}}\rangle$ can be described as follows:

$$|\psi''_{init}^{\text{Fanout}}\rangle = \epsilon_{K, \hat{X}}^{(K)} \epsilon_{M, \hat{X}}^{(M)} \epsilon_{O, \hat{X}}^{(O)} |\Psi^+\rangle_{KL} \otimes |\Psi^+\rangle_{MN} \otimes |\Psi^+\rangle_{OP}. \quad (45)$$

$$\sum_{S_{KL}}^{0,1} \sum_{S_{MN}}^{0,1} \sum_{S_{OP}}^{0,1} p_{S_{KL}} p_{S_{MN}} p_{S_{OP}} \hat{X}_K^{S_{KL}} \hat{X}_N^{S_{MN}} \hat{X}_P^{S_{OP}} (|0000\rangle + |1111\rangle)_{KLNP}. \quad (46)$$

Here, $|\psi_{final}^{\text{Fanout}}\rangle$ corresponds to the state in Eq. (3). Each $S_i = 1$ with probability p , otherwise it is 0. When all initial Bell pairs' fidelity are equally F , final state's fidelity becomes $F^3 - (1 - F)^3$.

4. Removal, Removal-Add

In Removal and Removal-Add operations, X errors on measured qubits do not change the measurement results. We describe these facts as follows:

$$\text{Rem}_{Q \rightarrow R} \epsilon_{Q, \hat{X}}^{(Q)} |\psi_{init}\rangle = |\psi_{final}\rangle, \quad (47)$$

$$\text{RemAdd}_{S \rightarrow T, U} \epsilon_{S, \hat{X}}^{(S)} |\psi_{init}\rangle = |\psi_{final}\rangle. \quad (48)$$

To conclude the above discussion we show the X error propagation in Fig. 10.

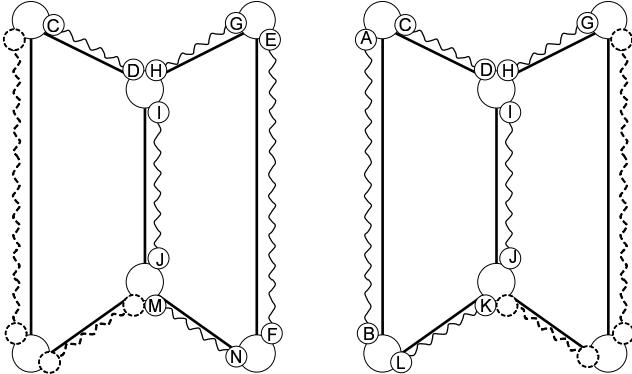


FIG. 10. X errors propagation. The left figure shows the five Bell pairs that affect on the final Bell pair AF. The right figure shows the five Bell pairs that affect on the final Bell pair BE.

5. Comparison

X error relations between the input states and the final state in the 2ES and QNC protocols can be described as follows: When we assume each Bell pair for initial resource has an X error on one qubit with probability

After Fanout, the final system $\epsilon_{K, \hat{X}}^{(K)} \epsilon_{N, \hat{X}}^{(M)} \epsilon_{P, \hat{X}}^{(O)} |\psi_{final}^{\text{Fanout}}\rangle$ becomes

$$P_{m,n} \text{ are as in Eq. 35, the initial state } |\psi''_{init}^{\text{QNC}}\rangle \text{ and final state } |\psi''_{final}^{\text{QNC}}\rangle \text{ become as follows:}$$

$$|\psi''_{init}^{\text{QNC}}\rangle = \epsilon_{M, \hat{X}}^{(M)} \epsilon_{K, \hat{X}}^{(K)} \epsilon_{I, \hat{X}}^{(I)} \epsilon_{G, \hat{X}}^{(G)} \epsilon_{E, \hat{X}}^{(E)} \epsilon_{C, \hat{X}}^{(C)} \epsilon_{A, \hat{X}}^{(A)} |\psi_{init}^{\text{QNC}}\rangle \quad (49)$$

$$|\psi''_{final}^{\text{QNC}}\rangle = \sum_m^{0,1} \sum_n^{0,1} P_{m,n} \hat{X}_A^m \hat{X}_B^n |\psi_{final}^{\text{QNC}}\rangle \quad (50)$$

$$= |\psi''_{final}^{\text{QNC}}\rangle \quad (51)$$

Thus, the final fidelities of the 2ES protocol with X or Z errors are the same. When we assume each Bell pair in our initial resource set has an X error on one qubit with probability p , the initial state $|\psi''_{init}^{\text{2ES}}\rangle$ and final state $|\psi''_{final}^{\text{2ES}}\rangle$ become as follows:

$$|\psi''_{init}^{\text{2ES}}\rangle = \epsilon_{M, \hat{X}}^{(M)} \epsilon_{I, \hat{X}}^{(I)} \epsilon_{C, \hat{X}}^{(C)} |\psi_{init}^{\text{2ES}}\rangle, \quad (52)$$

$$|\psi''_{final}^{\text{2ES}}\rangle = \sum_m^{0,1} P_m \hat{X}_A^m |\Psi^+\rangle_{CN} \\ = |\psi''_{final}^{\text{2ES}}\rangle. \quad (53)$$

Although the fidelity is the same, the location of errors which cause X or Z errors on the final Bell pairs are different. As a result, the relationship between input fidelity and output fidelity of our network coding protocol and 2-entanglement swapping are equal that of Z errors as shown in Fig. 8.

D. General Pauli error model

Finally, we model more general errors on our initial resource Bell pairs, as Pauli errors occurring during CNOT gates $\text{CNOT}_{\varepsilon}^{(control, target)}$ in the initial part of the total circuit in Fig. 13. We define the following errors ε on control and target qubits in every CNOT gate:

$$\text{CNOT}_{\varepsilon}^{(A, B)} |\psi_{input}^{\text{CNOT}}\rangle = \varepsilon_A \otimes \varepsilon_B |\psi_{output}^{\text{CNOT}}\rangle \quad (54)$$

$$\varepsilon_A \otimes \varepsilon_B = \sum_{i=0}^3 p_i \sigma_A^i \otimes \sum_{j=0}^3 p_j \sigma_B^j. \quad (55)$$

Here, $p_0 p_0 = 1 - p = F$ and $p_i p_j = \frac{p}{15}$ except for both $i = 0$ and $j = 0$. $\sigma^0, \dots, \sigma^3$ denote \hat{I} , \hat{X} , \hat{Y} , and \hat{Z} respectively.

We investigate the relation between the fidelity of the input states and that of our output state. Following the above setting, our initially shared seven Bell pairs $|\psi_{init}^{\varepsilon, \text{QNC}}\rangle$ include Pauli errors. Each Bell pair, which is a combination of sixteen possible error conditions, becomes a mixture of four states. For example, we describe the state of Bell pair AB below:

$$\mathbf{CNOT}_{\varepsilon}^{(A,B)} H_A |00\rangle_{AB} = \varepsilon_A \otimes \varepsilon_B |\Psi_{AB}^+\rangle \quad (56)$$

$$= \left(1 - \frac{4p}{5}\right) |\Psi_{AB}^+\rangle + \frac{4p}{15} |\Psi_{AB}^-\rangle \\ + \frac{4p}{15} |\Phi_{AB}^+\rangle + \frac{4p}{15} |\Phi_{AB}^-\rangle. \quad (57)$$

This expression arises because of the symmetric effect of some errors on Bell pairs, as in the following equations:

$$|\Psi_{AB}^+\rangle = (\hat{I}_A \otimes \hat{I}_B) |\Psi_{AB}^+\rangle = (\hat{X}_A \otimes \hat{X}_B) |\Psi_{AB}^+\rangle \\ = (\hat{Y}_A \otimes \hat{Y}_B) |\Psi_{AB}^+\rangle = (\hat{Z}_A \otimes \hat{Z}_B) |\Psi_{AB}^+\rangle, \quad (58)$$

$$|\Phi_{AB}^+\rangle = (\hat{I}_A \otimes \hat{X}_B) |\Psi_{AB}^+\rangle = (\hat{X}_A \otimes \hat{I}_B) |\Psi_{AB}^+\rangle \\ = (\hat{Y}_A \otimes \hat{Z}_B) |\Psi_{AB}^+\rangle = (\hat{Z}_A \otimes \hat{Y}_B) |\Psi_{AB}^+\rangle, \quad (59)$$

$$|\Psi_{AB}^-\rangle = (\hat{I}_A \otimes \hat{Z}_B) |\Psi_{AB}^+\rangle = (\hat{Z}_A \otimes \hat{I}_B) |\Psi_{AB}^+\rangle \\ = (\hat{X}_A \otimes \hat{Y}_B) |\Psi_{AB}^+\rangle = (\hat{Y}_A \otimes \hat{X}_B) |\Psi_{AB}^+\rangle, \quad (60)$$

$$|\Phi_{AB}^-\rangle = (\hat{X}_A \otimes \hat{Z}_B) |\Psi_{AB}^+\rangle = (\hat{Z}_A \otimes \hat{X}_B) |\Psi_{AB}^+\rangle \\ = (\hat{I}_A \otimes \hat{Y}_B) |\Psi_{AB}^+\rangle = (\hat{Y}_A \otimes \hat{I}_B) |\Psi_{AB}^+\rangle. \quad (61)$$

Based on the above, we assume all Pauli errors exist on the target qubits of CNOT gates in our initial resources. We show the relationship between errors on initial states and final state in Table II. For example, in the upper left corner of the table, the $\hat{I}_A \hat{X}_B$ entry indicates that an X error on the initial Bell pair AB results in an error-free Bell pair AF and an X error on the Bell pair BE , so that the final state is $|\Psi^+\rangle_{AF} |\Phi^+\rangle_{BE}$.

TABLE II. The relationship between errors on initial Bell pairs and final states. Columns correspond to the type of errors on underbarred qubits of initial Bell pairs.

Bell pair	\hat{X}	\hat{Y}	\hat{Z}
$ \Psi^+\rangle_{\underline{AB}}$	$\hat{I}_A \hat{X}_B$	$\hat{Z}_A \hat{X}_B$	$\hat{Z}_A \hat{I}_B$
$ \Psi^+\rangle_{\underline{CD}}$	$\hat{X}_F \hat{X}_B$	$\hat{Z}_A \hat{X}_F \hat{X}_B$	$\hat{Z}_A \hat{I}_B$
$ \Psi^+\rangle_{\underline{EF}}$	$\hat{X}_F \hat{I}_E$	$\hat{X}_F \hat{Z}_E$	$\hat{I}_F \hat{Z}_E$
$ \Psi^+\rangle_{\underline{GH}}$	$\hat{X}_F \hat{X}_B$	$\hat{X}_F \hat{X}_B \hat{Z}_E$	$\hat{I}_F \hat{Z}_E$
$ \Psi^+\rangle_{\underline{IJ}}$	$\hat{X}_F \hat{X}_B$	$\hat{Z}_A \hat{X}_F \hat{X}_B \hat{Z}_E$	$\hat{Z}_F \hat{Z}_E$
$ \Psi^+\rangle_{\underline{KL}}$	$\hat{I}_A \hat{X}_B$	$\hat{Z}_A \hat{X}_B \hat{Z}_E$	$\hat{Z}_A \hat{Z}_E$
$ \Psi^+\rangle_{\underline{MN}}$	$\hat{X}_F \hat{I}_B$	$\hat{Z}_A \hat{X}_F \hat{Z}_B$	$\hat{Z}_A \hat{Z}_B$

We show the relationship between the input fidelity and the output fidelity of our network coding protocol and 2-entanglement swapping in Fig. 11. Here, the final state with $F_{output} < 0.5$ has no practical use for quantum communication. When $F_{input} \leq 0.88$, the 2ES protocol falls below $F_{out} = 0.5$. When $F_{input} \leq 0.9$, the QNC protocol also falls below $F_{out} = 0.5$.

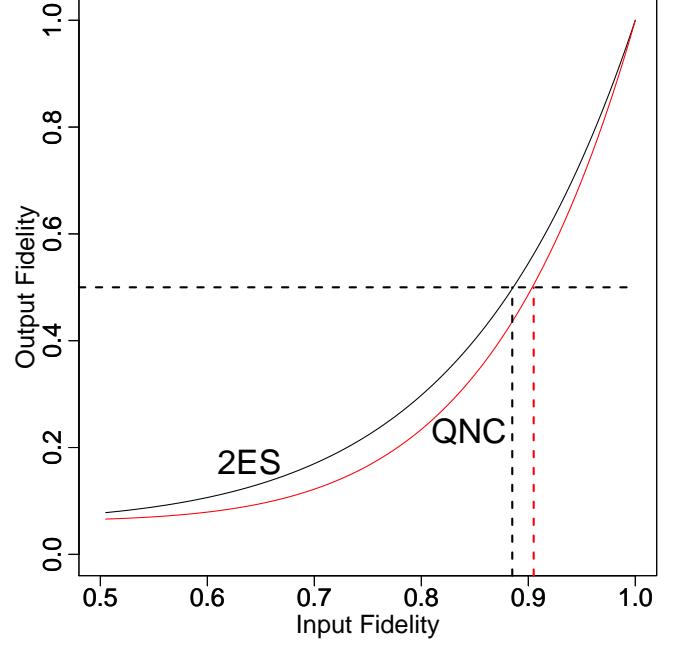


FIG. 11. Joint fidelity of the two output Bell pairs. We compare Swapping and QNC with general Pauli error model. Both show a substantial penalty compared to the fidelity of a single Bell pair.

IV. INCORPORATING GATE ERRORS

In this section, we investigate the error propagation caused by local gates in each encoding step which shown in Fig. 13. We introduce $\mathbf{Con}_{\varepsilon}$, $\mathbf{Add}_{\varepsilon}$, $\mathbf{Fanout}_{\varepsilon}$, and $\mathbf{QNC}_{\varepsilon}$. These operators use $\mathbf{CNOT}_{\varepsilon}$ within these operations. Furthermore, the following error ϵ occur on all qubits in every measurement, single qubit gate, and waiting time.

$$\epsilon = \sum_{i=0}^3 p_i \sigma^i \quad (62)$$

Here, $p_0 = F$ and $p_i = \frac{p}{3}$ whenever $i \neq 0$. In subsections IV A through IV E, we give a step by step qualitative analysis, then in subsection IV F we present the results of our Monte Carlo simulation of the complete circuit.

A. Errors in Step 1

In step 1, the CNOT gate in Connection causes the following errors $\varepsilon^{(1)}$:

$$\mathbf{Con}_{\varepsilon, G \rightarrow H}^E \mathbf{Con}_{\varepsilon, C \rightarrow D}^A |\psi_{init}\rangle = \varepsilon^{(1)} |\psi_{final}\rangle. \quad (63)$$

When we assume the initial resources and CNOT gates in other steps do not include errors, we can describe the relationship between errors in this step and final states as shown in Table III.

TABLE III. The relationship between errors caused by CNOT gates in Step 1 and final states. Columns correspond to the type of errors on underlined qubits.

Qubit(underlined)	\hat{X}	\hat{Y}	\hat{Z}
$\mathbf{CNOT}^{(\underline{A}, \underline{C})}$	\hat{X}_A	\hat{Y}_A	\hat{Z}_A
$\mathbf{CNOT}^{(\underline{A}, \underline{C})}$	$\hat{X}_B \hat{X}_F$	$\hat{X}_B \hat{X}_F$	\hat{I}
$\mathbf{CNOT}^{(\underline{E}, \underline{G})}$	\hat{X}_E	\hat{Y}_E	\hat{Z}_E
$\mathbf{CNOT}^{(\underline{E}, \underline{G})}$	$\hat{X}_B \hat{X}_F$	$\hat{X}_B \hat{X}_F$	\hat{I}

B. Errors in Step 2.

In step 2, the CNOT gate in Add causes the following errors $\varepsilon^{(2)}$:

$$\mathbf{Add}_{\varepsilon, I \rightarrow J}^{D, H} |\psi_{(1)}\rangle = \varepsilon^{(2)} |\psi_{final}\rangle. \quad (64)$$

When we assume the initial resources and CNOT gates in other steps do not include errors, we can describe the relationship between errors in this step and final states as shown in Table IV.

TABLE IV. The relationship between errors caused by CNOT gates in Step 2 and final states. Columns correspond to the type of errors on underlined qubits.

Qubit(underlined)	\hat{X}	\hat{Y}	\hat{Z}
$\mathbf{CNOT}^{(\underline{D}, \underline{I})}$	\hat{I}	\hat{Z}_A	\hat{Z}_A
$\mathbf{CNOT}^{(\underline{D}, \underline{I})}$	$\hat{X}_B \hat{X}_F$	$\hat{X}_B \hat{Z}_E \hat{X}_F$	\hat{Z}_E
$\mathbf{CNOT}^{(\underline{H}, \underline{I})}$	\hat{I}	\hat{Z}_E	\hat{Z}_E
$\mathbf{CNOT}^{(\underline{H}, \underline{I})}$	$\hat{X}_B \hat{X}_F$	$\hat{X}_B \hat{X}_F$	\hat{I}

C. Errors in Step 3.

In step 3, the CNOT gate in Fanout causes the following errors $\varepsilon^{(3)}$:

$$\mathbf{Fanout}_{\varepsilon, K \rightarrow L, M \rightarrow N}^J |\psi_{(2)}\rangle = \varepsilon^{(3)} |\psi_{final}\rangle. \quad (65)$$

When we assume the initial resources and CNOT gates in other steps do not include errors, we can describe the relationship between errors in this step and final states as shown in Table V.

TABLE V. The relationship between errors caused by CNOT gates in Step 3 and final states. Columns correspond to the type of errors on underlined qubits.

Qubit(underlined)	\hat{X}	\hat{Y}	\hat{Z}
$\mathbf{CNOT}^{(\underline{J}, \underline{K})}$	\hat{X}_F	$\hat{Z}_A \hat{Z}_E \hat{X}_F$	$\hat{Z}_A \hat{Z}_E$
$\mathbf{CNOT}^{(\underline{J}, \underline{K})}$	\hat{X}_B	\hat{X}_B	\hat{I}
$\mathbf{CNOT}^{(\underline{J}, \underline{M})}$	\hat{I}	$\hat{Z}_A \hat{Z}_E$	$\hat{Z}_A \hat{Z}_E$
$\mathbf{CNOT}^{(\underline{J}, \underline{M})}$	\hat{X}_F	\hat{X}_F	\hat{I}

TABLE VI. The relationship between errors caused by CNOT gates in Step 4 and final states. Columns correspond to the type of errors on underlined qubits.

Qubit(underlined)	\hat{X}	\hat{Y}	\hat{Z}
$\mathbf{CNOT}^{(\underline{L}, \underline{B})}$	\hat{I}	$\hat{Z}_A \hat{Z}_E$	$\hat{Z}_A \hat{Z}_E$
$\mathbf{CNOT}^{(\underline{L}, \underline{B})}$	\hat{X}_B	\hat{Y}_B	\hat{Z}_B
$\mathbf{CNOT}^{(\underline{N}, \underline{F})}$	\hat{I}	$\hat{Z}_A \hat{Z}_E$	$\hat{Z}_A \hat{Z}_E$
$\mathbf{CNOT}^{(\underline{N}, \underline{F})}$	\hat{X}_F	\hat{Y}_F	\hat{Z}_F

D. Errors in Step 4.

In step 4, the CNOT gate operations cause the following errors $\varepsilon^{(4)}$:

$$\mathbf{CNOT}_{\varepsilon}^{(N, F)} \mathbf{CNOT}_{\varepsilon}^{(L, B)} |\psi_{(3)}\rangle = \varepsilon^{(4)} |\psi_{final}\rangle. \quad (66)$$

When we assume the initial resources and CNOT gates in other steps do not include errors, we can describe the relationship between errors in this step and final states as shown in Table VI.

E. Errors in Step 5-7.

In these steps, no additional errors are added to the system.

F. Simulations for total errors

Using these results, the final state can be described as follows:

$$\mathbf{QNC}_{\varepsilon}^{''''} |\psi_{final}^{\mathbf{QNC}}\rangle = \varepsilon^{(4)} \varepsilon^{(3)} \varepsilon^{(2)} \varepsilon^{(1)} \varepsilon^{init} |\psi_{final}\rangle \quad (67)$$

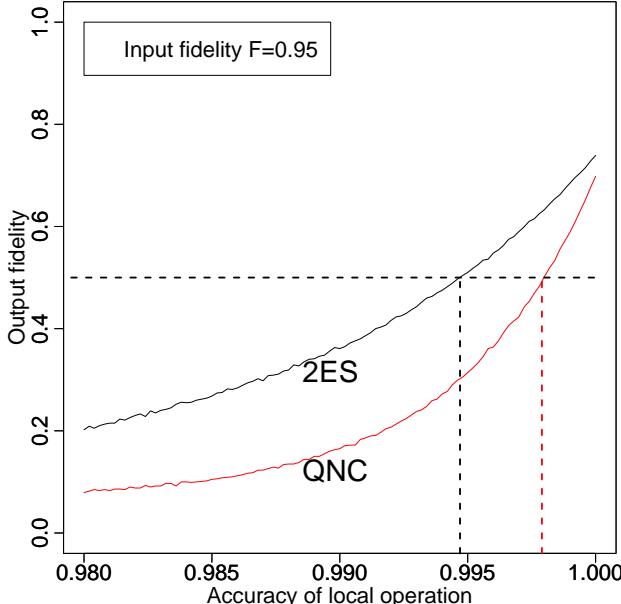
$$= |\psi_{final}^{''''}\rangle \quad (68)$$

Then, we show the relation between the input fidelity of Bell pairs, the accuracy of local operations, and the output fidelity in Fig. 12.

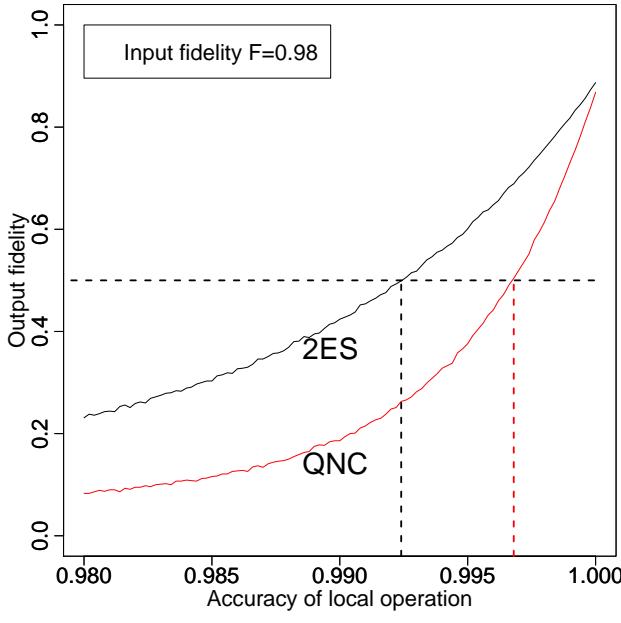
To calculate these fidelities, we used Monte Carlo simulations. In each simulation, the fidelities of Bell pairs are fixed to $F = 0.95$ or $F = 0.98$. The accuracy of local operations is changed from $F = 0.980$ to $F = 1.000$ using $\Delta F = 0.001$. In each parameter set, the simulation until we accumulate twenty thousand errors on the final states (up to a maximum of one hundred million times.).

V. CONCLUSION

We have shown the propagation of errors in quantum network coding protocols using the example of the butterfly network. We also show the error threshold of quantum network coding in noisy quantum repeater networks using Monte-Carlo simulations. We can see that QNC is more sensitive to local gate errors than entanglement



(a)



(b)

FIG. 12. Comparison of Swapping and QNC with incorporating gate errors. Output fidelities correspond to the case with no error on either final Bell pair. (a) Initial fidelity $F = 0.95$. (b) Initial fidelity $F = 0.98$.

swapping. In the case of the butterfly network, 2ES tolerates about twice the local error rate of QNC. From these results, we see that each scheme is suitable for different purposes. 2ES is useful when the quantum resources are abundant or low communication speed is permitted. Quantum network coding is useful when the quantum resources are limited or high communication speed is required. The choice of scheme therefore depends on the environment of the quantum network and the quantum application used. We hope quantum network coding will be used in actual future repeater networks.

VI. ACKNOWLEDGEMENTS

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[1] S. Lloyd, J. H. Shapiro, F. N. C. Wong, P. Kumar, S. M. Shahriar, and H. P. Yuen, SIGCOMM Comput. Commun. Rev. **34**, 9 (2004).

[2] H. J. Kimble, Nature **453**, 1023 (2008).

[3] R. Van Meter, *Quantum Networking* (Wiley-ISTE, 2014).

[4] C. H. Bennett and G. Brassard, in *Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing*, Vol. 11 (1984) pp. 175–179.

[5] A. Ekert, Phys. Rev. Lett. **67**, 661 (1991).

[6] W. K. Wootters and W. H. Zurek, Nature **299**, 802 (1982).

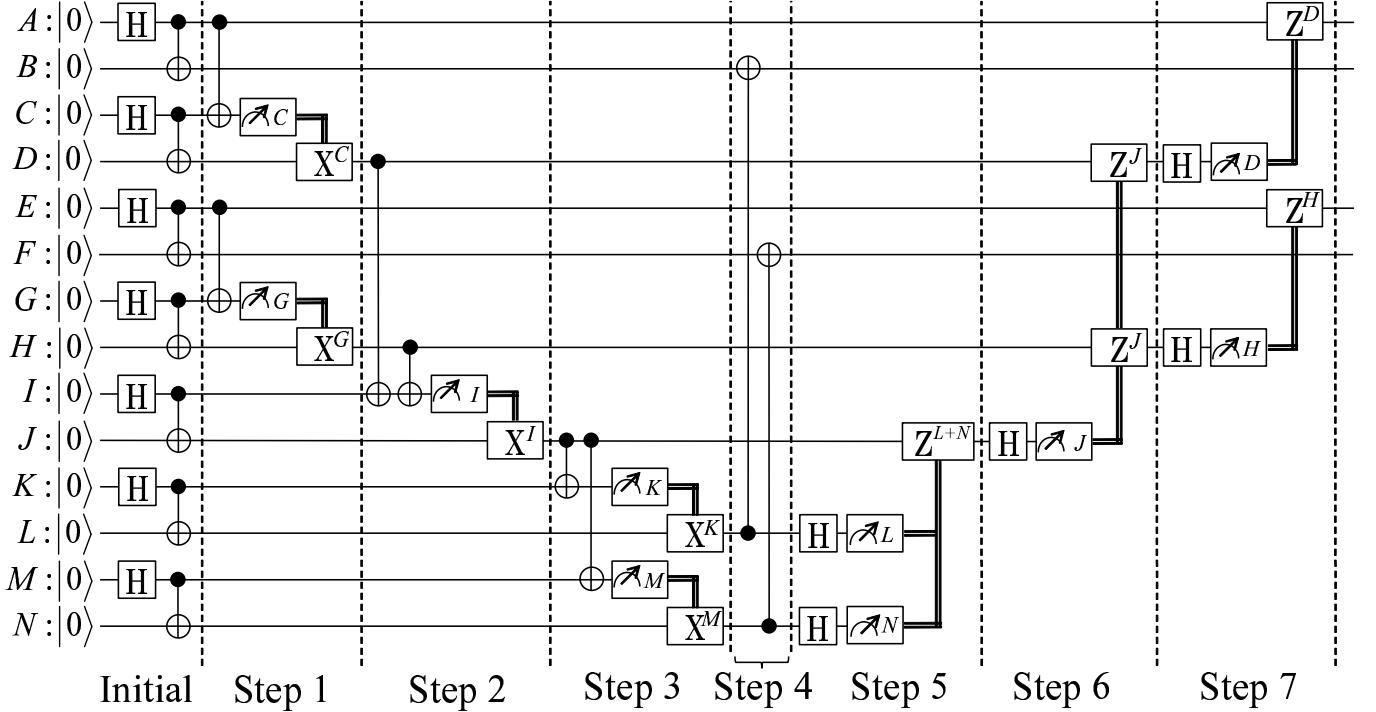


FIG. 13. Complete circuit for QNC. Numbers refer to the step of QNC procedure. The initial Bell pair creation is modeled as a Hadamard gate followed by a CNOT, with a separate error probability from the rest of the circuit.

- [7] M. Peev, et al, *New Journal of Physics* **11**, 075001 (2009).
- [8] M. Sasaki, et al, *Opt. Express* **19**, 10387 (2011).
- [9] M. Ben-Or and A. Hassidim, in *Proceedings of the thirty-seventh annual ACM symposium on Theory of computing* (ACM, 2005) pp. 481–485.
- [10] A. Broadbent, J. Fitzsimons, and E. Kashefi, in *Formal Methods for Quantitative Aspects of Programming* (Springer, 2010) pp. 43–86.
- [11] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **81**, 5932 (1998).
- [12] L.-M. Duan and C. Monroe, *Rev. Mod. Phys.* **82**, 1209 (2010).
- [13] D. Huel, I. Inlek, G. Vittorini, C. Crocker, S. Debnath, S. Clark, and C. Monroe, *Nature Physics* **11**, 37 (2014).
- [14] W. Dür and H. J. Briegel, *Reports on Progress in Physics* **70**, 1381 (2007).
- [15] R. Van Meter, T. D. Ladd, W. J. Munro, and K. Nemoto, *IEEE/ACM Trans. Netw.* **17**, 1002 (2009).
- [16] L. Jiang, J. M. Taylor, K. Nemoto, W. J. Munro, R. Van Meter, and M. D. Lukin, *Phys. Rev. A* **79**, 032325 (2009).
- [17] W. J. Munro, K. A. Harrison, A. M. Stephens, S. J. Devitt, and K. Nemoto, *Nature Photonics* **4**, 792 (2010).
- [18] A. G. Fowler, D. S. Wang, C. D. Hill, T. D. Ladd, R. Van Meter, and L. C. L. Hollenberg, *Phys. Rev. Lett.* **104**, 180503 (2010).
- [19] M. Żukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, *Phys. Rev. Lett.* **71**, 4287 (1993).
- [20] R. Ahlswede, N. Cai, S. R. Li, and R. W. Yeung, *IEEE Transactions on Information Theory* **46**, 1204 (2000).
- [21] M. Hayashi, K. Iwama, H. Nishimura, R. Raymond, and S. Yamashita, in *Symposium on Theoretical Aspects of Computer Science* (2007) pp. 610–621.
- [22] D. Leung, J. Oppenheim, and A. Winter, *IEEE Transactions on Information Theory* **56**, 3478 (2010).
- [23] M. Hayashi, *Phys. Rev. A* **76**, 040301 (2007).
- [24] Y. Shi and E. Soljanin, in *40th Annual Conference on Information Sciences and Systems* (2006) pp. 871–876.
- [25] H. Kobayashi, F. Le Gall, H. Nishimura, and M. Roetteler, in *36th International Colloquium on Automata, Languages and Programming* (2009) pp. 622–633.
- [26] H. Kobayashi, F. Le Gall, H. Nishimura, and M. Roetteler, in *2010 IEEE International Symposium on Information Theory* (2010) pp. 2686–2690.
- [27] H. Kobayashi, F. Le Gall, H. Nishimura, and M. Roetteler, in *2011 IEEE International Symposium on Information Theory* (2011) pp. 109–113.
- [28] T. Satoh, F. Le Gall, and H. Imai, *Phys. Rev. A* **86**, 032331 (2012).
- [29] L. Aparicio and R. Van Meter, in *Proc. SPIE*, Vol. 8163 (2011) p. 816308.